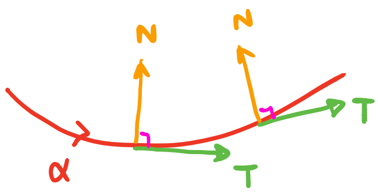
§ Gauss map & Second Fundamental Form

We now study the extrinsic geometry of surfaces and define various notions of curvatures for surfaces.

Recall : (Plane curves)



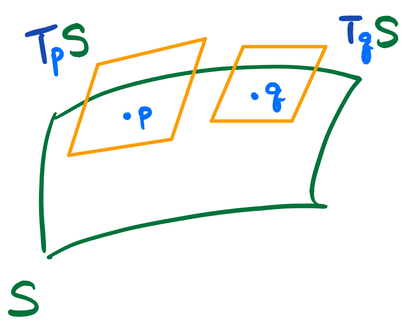
Frenet eq$^{\underline{n}}$: $\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$

$\Downarrow$

$$\left[\text{Curvature } k = \text{rate of change of } T = -(\text{rate of change of } N) \right]$$

Now, for an (orientable) surface S, we consider its

tangent bundle TS "=" a family of tangent planes T_pS



HOPE :

"Curvature of S" = rate of change of T_pS

= − rate of change of unit normal N_p

Note : In $\mathbb{R}^3$, a 2-dim'l subspace P is determined uniquely (up to a sign) by its unit normal $N \perp P$.

Def$^{\underline{n}}$: Let $S \subseteq \mathbb{R}^3$ be an orientable surface, oriented by a global unit normal vector field called

$$N : S \longrightarrow \underset{\substack{\uparrow \\ \text{unit sphere} \\ \text{in } \mathbb{R}^3}}{S^2}$$

Gauss map

The Gauss map N is a smooth map from S to S^2

$\Rightarrow$ we can consider its differential at any $p \in S$

$$dN_p : T_pS \xrightarrow{\text{linear}} T_{N(p)}S^2 \cong N(p)^{\perp} = T_pS$$

Def$^{\underline{n}}$: The shape operator / Weingarten map (at p) is the linear operator on T_pS defined by

$$\mathcal{S} = -dN_p : T_pS \xrightarrow{\text{linear}} T_pS$$

Def$^{\underline{n}}$:

$$\boxed{H := \operatorname{tr} \mathcal{S}} \leftarrow \text{mean curvature}$$

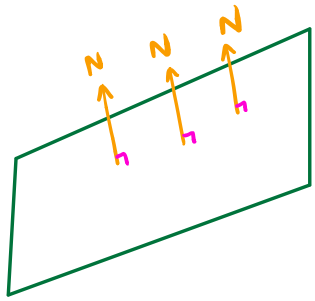
$$\boxed{K := \det \mathcal{S}} \leftarrow \text{Gauss curvature}$$

Effect of orientation:

$$N \rightsquigarrow -N \;\Rightarrow\; \mathcal{S} \rightsquigarrow -\mathcal{S} \;\Rightarrow\; \begin{array}{l} H \rightsquigarrow -H \\ \underbrace{K \rightsquigarrow K}_{\text{unchanged !!}} \end{array}$$

Examples:

(1) Planes



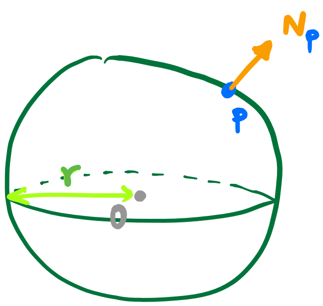
$N \equiv$ const. vector

$$\Rightarrow \quad S = -dN = 0 \qquad \text{i.e.} \quad S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\begin{array}{l} H \equiv 0 \\ K \equiv 0 \end{array}} \qquad \text{"flat"}$$

(2) Spheres

$$S = \mathbb{S}^2(r) = \{ p \in \mathbb{R}^3 : \|p\| = r \}$$



$S = \mathbb{S}^2(r)$

$$N(p) = \frac{p}{\|p\|} = \frac{1}{r} p$$

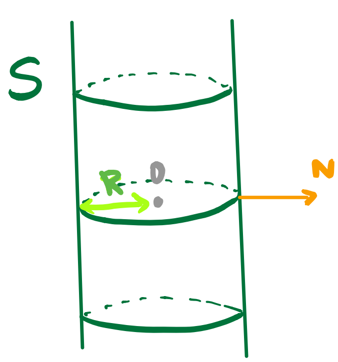
$$\Rightarrow \quad S = -dN = -\frac{1}{r} \mathrm{Id}, \quad \text{i.e.} \quad S = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\begin{array}{l} H \equiv -\frac{2}{r} \\ K \equiv \frac{1}{r^2} \end{array}}$$

Constant mean & Gauss curvature

(3) Cylinder

$$S = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = R^2 \}$$



$$N(x,y,z) = \frac{1}{R}(x,y,0)$$

Question: How to compute

$$S = -dN \; ?$$

Ans: Use local coordinates!

Locally, we can parametrize the cylinder by "cylindrical coordinates"

$$X(\theta, z) := (R\cos\theta,\ R\sin\theta,\ z)$$

At each point on S, T_pS is spanned by

$$\begin{cases} \dfrac{\partial}{\partial\theta} := \dfrac{\partial X}{\partial\theta} = (-R\sin\theta,\ R\cos\theta,\ 0) \\ \dfrac{\partial}{\partial z} := \dfrac{\partial X}{\partial z} = (0, 0, 1) \end{cases}$$

Hence, a (local) unit normal vector field is

$$N(\theta, z) = \frac{\frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z}}{\left\| \frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z} \right\|} = (\cos\theta, \sin\theta, 0)$$

$$\Rightarrow \begin{cases} dN\left(\frac{\partial}{\partial\theta}\right) = \dfrac{\partial N}{\partial\theta} = (-\sin\theta,\ \cos\theta,\ 0) \\ dN\left(\frac{\partial}{\partial z}\right) = \dfrac{\partial N}{\partial z} = (0, 0, 0) \end{cases}$$

In terms of the basis $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\}$ for T_pS

$$S = -dN = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{H \equiv -\frac{1}{R}, \quad K \equiv 0}$$

Note: In all the examples above, S are represented by symmetric matrices. This is in fact a general phenomena.

Prop: $S = -dN_p : T_pS \to T_pS$ is a self adjoint operator on the inner product space $(T_pS, \langle , \rangle)$.

Proof: Take any parametrization $\mathbf{X}(u,v)$ near p

$$T_pS = \operatorname{span}\left\{ \frac{\partial \mathbf{X}}{\partial u}, \frac{\partial \mathbf{X}}{\partial v} \right\}$$

It suffices to prove

$$\boxed{\left\langle S\left(\frac{\partial \mathbf{X}}{\partial u}\right), \frac{\partial \mathbf{X}}{\partial v} \right\rangle = \left\langle \frac{\partial \mathbf{X}}{\partial u}, S\left(\frac{\partial \mathbf{X}}{\partial v}\right) \right\rangle} \qquad (*)$$

By abuse of notation, we write

$$N(u,v) := N \circ \mathbf{X}(u,v)$$

Since $N_p \perp T_pS$ for all $p \in S$,

$$\left\langle N, \frac{\partial \mathbf{X}}{\partial v} \right\rangle \equiv 0$$

$$\xRightarrow[\text{w.r.t. } u]{\text{differentiate}} \quad \left\langle \underbrace{\frac{\partial N}{\partial u}}_{dN\left(\frac{\partial \mathbf{X}}{\partial u}\right) = -S\left(\frac{\partial \mathbf{X}}{\partial u}\right)}, \frac{\partial \mathbf{X}}{\partial v} \right\rangle + \left\langle N, \frac{\partial^2 \mathbf{X}}{\partial u \partial v} \right\rangle \equiv 0$$

Hence, $\langle S(\frac{\partial X}{\partial u}), \frac{\partial X}{\partial v} \rangle = \langle N, \frac{\partial^2 X}{\partial u \partial v} \rangle$

Similarly, $\langle N, \frac{\partial X}{\partial u} \rangle \equiv 0$

$$\xRightarrow[\text{w.r.t. } v]{\text{differentiate}} \langle S(\frac{\partial X}{\partial v}), \frac{\partial X}{\partial u} \rangle = \langle N, \frac{\partial^2 X}{\partial v \partial u} \rangle$$

$$\boxed{\frac{\partial^2 X}{\partial u \partial v} = \frac{\partial^2 X}{\partial v \partial u}} \Longrightarrow (*)$$

"mixed partials are the same"

$\square$

Def$^{\underline{n}}$: The second fundamental form (at p) with respect to N is the symmetric bilinear form

$$A : T_pS \times T_pS \longrightarrow \mathbb{R}$$

$$\boxed{A(u,v) := \langle Su, v \rangle}$$

Note: In local coord., $T_pS = \text{span}\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\}$

$$A = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{\text{symmetric matrix}} \quad \text{where } A_{ij} := A(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j})$$

By Spectral Theorem, the self adjoint operator

$$S = -dN_p : T_pS \longrightarrow T_pS \qquad \text{diagonalizable}$$

eigenvalues:	κ_1 ,	κ_2	principal curvatures	(at p)
eigenvectors (unit):	v_1 ,	v_2	principal directions	

<u>Def$^{\underline{n}}$</u>: $p \in S$ is an umbilic point $\iff$ $\boxed{\kappa_1 = \kappa_2}$ at p

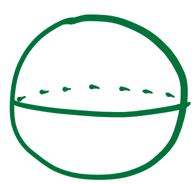
S is totally umbilic $\iff$ every $p \in S$ is umbilic.

<u>Examples</u>:

<u>Plane</u>

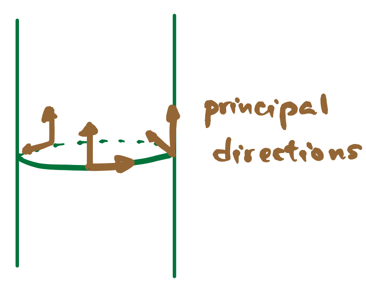
$\kappa_1 = \kappa_2 \equiv 0$

<u>Sphere</u>



$\kappa_1 = \kappa_2 \equiv -\frac{1}{R}$

<u>Cylinder</u>



$\kappa_1 = -\frac{1}{R} \quad \kappa_2 = 0$

totally umbilic

not umbilic